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# Special concept of a point-like nucleus with supercritical charge

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**Abstract**
The Dirac equation for an electron in the central Coulomb field of a point-like and almost point-like nucleus with the charge greater than 137 is considered. This singular problem, to which the fall-down onto the centre is inherent, is addressed using a new approach, based on a special concept of the singular centre, capable of producing results independent of the nucleus size. To this end, the Dirac equation is presented as a generalized eigenvalue boundary problem of a self-adjoint operator. The eigenfunctions make complete sets, orthogonal with a singular measure, and describe particles, asymptotically free and delta-function-normalizable both at infinity and near the singular centre $r = 0$. The barrier transmission coefficient for these particles responsible for the effects of electron absorption and spontaneous electron–positron pair production is found analytically as a function of electron energy and charge of the nucleus. The singular threshold behaviour of the corresponding amplitudes substitutes for the resonance behaviour, typical of the conventional theory that depends on the cut-off procedure at the edge of a finite-size nucleus.

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## 1. Introduction

The radial Dirac equation for an electron in the Coulomb potential

$$-\alpha Z/r, \tag{1}$$

of a point-like nucleus, where $\alpha$ is the fine-structure constant, $\alpha = 1/137$, and $Z$ is the nucleus charge (for formal purposes negative values of $Z$ will also be included into our consideration), and is a set of two first-order differential equations [1, 2]. It is convenient to write it in the following matrix form:

$$\mathcal{L}\Psi(r) = \left(\varepsilon + \frac{Z\alpha}{r}\right)\Psi(r), \qquad r \in (0, \infty), \tag{2}$$

$$\mathcal{L} = -\mathrm{i}\sigma_2 \frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\sigma_1 + m\sigma_3. \tag{3}$$

Here, the wavefunction is the two-component spinor

$$\Psi(r) = \begin{pmatrix} G(r) \\ F(r) \end{pmatrix}, \tag{4}$$

$\sigma_i$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

$m$ is the electron mass and $\varepsilon$ is its energy. The quantity $\kappa$ is the orbital momentum

$$\begin{aligned} \kappa &= -(l+1) && \text{for} \quad j = l + \tfrac{1}{2}, \\ \kappa &= l && \text{for} \quad j = l - \tfrac{1}{2}. \end{aligned} \tag{6}$$

In what follows we confine ourselves to the lowest orbital state $\kappa = -1$. Solutions to equation (2) are known [1, 2] in terms of confluent hypergeometric functions.

We are most interested in the supercritical case $\alpha|Z| > 1$, of which the falling to the centre is characteristic. One can exclude one component from (2) to reduce it to a second-order differential equation (see, e.g., [1–4]). Then the Coulomb term (1) in (2) gives rise to the inverse-radius-squared singular term $-(\alpha Z/r)^2$ in the potential, which is negative (attraction) irrespective of the sign of $\alpha Z$. This illustrates the presence of the fall-down onto the centre phenomenon inherent to the Dirac equation (2) with $\alpha|Z| > 1$. This case is of direct physical importance as introducing the effect [5–7, 3, 8] of spontaneous electron–positron pair creation by an overcharged nucleus created for a short time in heavy-ion collisions (see the review [9]) and also the effect of strong absorption of electrons by this nucleus—see our previous paper [10]. (Attribution of absorbing quality to a singular centre in other problems and within other approaches may be found in [11, 12].) Unlike [10], we find it now more appropriate to be dealing directly with the set of two first-order differential equations (2).

To handle the problem, with the supercritical case included, we define $\mathcal{L}$(3) as a self-adjoint operator, with the Dirac equation (2) presenting a generalized eigenvalue problem for it, and identify physical states with its eigenvectors. We stress that $\mathcal{L}$ is not the Hamiltonian, defined as $\mathcal{H} = \mathcal{L} - \frac{\alpha Z}{r}$. There are many ways to present a given differential equation as a generalized eigenvalue problem for different operators. Our choice (3) of $\mathcal{L}$ is mainly dictated by the demand that the term, responsible for the falling to the centre, be kept in the right-hand side in equation (2), to later show itself as a singularity of the measure, serving the scalar products in the space, spanned by solutions of the generalized eigenvalue problem. In other words, our procedure partially extracts the singularity from the interaction and places it into the measure. The singularity in the measure gives the possibility to treat the solutions of the Dirac equation oscillating near the origin as particles, *free* in this vicinity. These are $\delta$-function-normalizable and belong to the *continuum* of eigenvectors of $\mathcal{L}$. This offers the physical concept of the singular centre as an object, emitting and absorbing particles, reminiscent of a black hole.

This is different from what may be obtained using the self-adjoint extension of the Hamiltonian within the von Neuman technique, resulting in a discrete spectrum, unlimited from below, for the Schrödinger equation [4, 13], and in a discrete spectrum, condensing towards the infinitely strong binding point $\varepsilon = -m$, as it can be established by inspecting the equation for the Dirac spectrum, found in [4]. For this reason, there is no place for the spontaneous pair creation effect in that approach.

The present work is an extension to the Dirac equation of analogous treatment, developed earlier [14, 15] for the (second-order) Schrödinger equation with singular potential with inverse-square singularity. We find that the extension to the relativistic case is straightforward, with the only complication due to the spinorial character of the Dirac equation. This is the lack of positive definiteness of the measure in the definition of scalar products for negative energy and positive charge. It is shown, however, that one can easily restrict oneself to vectors with a definite sign of the norm within one set, and also exclude zero-norm vectors by imposing a certain easy-to-observe convention concerning the way the ends of the interval $(r_L, r_U)$, where equation (2) is defined—first taken finite—tend to their final positions in the singularity points $r = 0$ and $r = \infty$.

In section 2, we set the generalized self-adjoint eigenvalue problem associated with the Dirac equation in a two-side-limited box $r_L < r < r_U, r_L \to 0, r_U \to \infty$ with zero and (anti)periodic boundary conditions, and derive the orthogonality relations with a singular measure and point a family, labelled by the ratio $R = \alpha Z/\varepsilon$, of Hilbert spaces, spanned by solutions of the generalized eigenvalue problem, including the ones that are $\delta$-function-normalizable and correspond to particles, free both near the origin $r = 0$ and near $r = \infty$. In section 3, we describe the coordinate transformation $\xi(r)$ that reduces the generalized eigenvalue problem to the standard one by mapping the origin $r = 0$ to infinitely remote point $\xi = \pm\infty$ (depending on the sign of the energy). In the new coordinate $\xi$, the free character of the wavefunction behaviour near the singular point $r = 0$ becomes explicit, and its normalization to $\delta$-function becomes standard. Referring to the one-dimensional barrier problem on the infinite axis $\xi$, to which the Dirac equation is reduced in our procedure, we find in section 4 the reflection and transmission coefficients for the waves, reflected from and transmitted to the singularity in $r = 0$. It is important that, in the Dirac equation, in contrast to the Klein–Gordon case, the function, inverse to the transformation $\xi(r)$, is two-valued in the negative-energy continuum. This fact leads to the change of identification of the incident and reflected waves near $r = 0$ as compared to the positive-energy domain, and results in the absence of the superradiation phenomenon, described in [16, 17]. In section 5, the transmission coefficient is interpreted within the Dirac sea picture as probability of absorption of electrons by the supercritical nucleus in the positive-energy range, and as distribution of spontaneously produced electrons—which outgo to the singular point $r = 0$, and positrons—which outgo to the singular point $r = \infty$. These coefficients are studied in detail and plotted as functions of the nucleus charge and electron energy. Their singular threshold behaviour near the point $\alpha^2 Z^2 = 1$ is established. In concluding section 6, we comment on a comparison with the known results [3, 8] about the spontaneous pair creation, obtained using the finite nucleus-size cut-off, and discuss the lines, along which the second-quantized theory of unstable vacuum [17–19] could be applied to the supercritical nucleus, based on the consideration made in the present paper.

## 2. Bliss eigenvalue problem

There are two singular points of the set of differential equations (2), $r = \infty$ and $r = 0$.

The fundamental solutions behave near the point $r = \infty$ as

$$e^{\mp i p r} r^{\mp i\zeta}, \tag{7}$$

where $p = \sqrt{\varepsilon^2 - m^2}$, $\zeta = \frac{\alpha Z \varepsilon}{p}$.

It will be convenient for our present purposes to present the well-known facts about this singular point in the following words .

When $|\varepsilon| < m$, the growing solution should be disregarded, whereas the other decreases and is thus localized in the finite region of the $r$-space, far from the singularity point $r = \infty$.

We refer to this situation by saying: the singularity in the infinitely remote point repulses the particle.

When $|\varepsilon| > m$, the both solutions (7) oscillate. They are $\delta$-function-normalizable, because they are concentrated mostly near the infinitely remote singular point. We refer to this situation by saying that this singularity attracts particles. The particles, captured by the infinitely remote singularity are asymptotically free in the sense that the fundamental solutions and the spectrum do not depend on $Z, \kappa$ (if it is not fixed), in other words, on any terms in (2) that do not survive in the limit $r \to \infty$, as compared to the constant term $\varepsilon$, responsible for the singularity in the infinity. The solutions with positive and negative signs in the exponentials in (7), the incoming and outgoing waves in customary wording, may be thought of as ones, emitted and absorbed by the infinitely remote singularity.

Now we turn to the other singular point of the set of differential equations (2), the one in the origin $r = 0$, and shall treat it in exactly the same manner. Two fundamental solutions to equation (2) behave near the origin $r \to 0$ like

$$r^{\pm \mathrm{i}\gamma}, \tag{8}$$

where $\gamma = \sqrt{\alpha^2 Z^2 - 1}$.

When $\alpha|Z| < 1$, the growing solution should be disregarded, whereas the other decreases as $r \to 0$ and is thus localized far from the singularity point. We refer to this situation by saying: the singularity in the origin repulses the particle.

When $\alpha|Z| > 1$ (supercritical charge), the both solutions (8) oscillate, which is typical of the problems with singular attractive potential. Within a generalized eigenvalue problem for the operator associated with the differential expression $\mathcal{L}$ (3), formulated below, they are $\delta$-function-normalizable, because the measure characteristic of this problem appears to be singular in the origin, this fact providing the necessary divergence of the norm. The particle is thus localized mostly near the origin, attracted by the singularity. The particles, attracted by the singular centre, are asymptotically free in its vicinity in the sense that the fundamental solutions and the spectrum do not depend on $\varepsilon, m$, in other words, on any terms in (2) that do not survive in the limit $r \to 0$, as compared to the singular potential. The solutions with positive and negative signs in the exponentials (8), may be thought of as waves, emitted and absorbed by the singular centre.

With the ratio

$$R = \frac{\alpha Z}{\varepsilon} \tag{9}$$

fixed, equation (2) can be written either as

$$\mathcal{L}\Psi(r) = \varepsilon \left(1 + \frac{R}{r}\right) \Psi(r) \tag{10}$$

or as

$$\mathcal{L}\Psi(r) = \alpha Z \left(\frac{1}{R} + \frac{1}{r}\right) \Psi(r). \tag{11}$$

Thus, equation (2) becomes a generalized eigenvalue problem of the type where the eigenvalue (this is either $\varepsilon$ for (10) or $\alpha Z$ for (11)) is multiplied by a function of the variable $r$ and not by unity. In the case of a second-order equation, an analogous eigenvalue problem was first studied by Kamke [20]—we called it Kamke eigenvalue problem when using it as applied to the singular Schrödinger equation in [15, 10], but for the set like (2) it was considered, according to [20], much earlier by Bliss [21].

The equation, in which the sign in front of the derivative term is reversed and, besides, all the matrices are transposed, is called an *adjoint* equation. Equation (2) is *self-adjoint* in the sense that solutions $\overline{\Psi}(r)$ of its adjoint equation are linearly expressed in terms of its solutions with the help of a nondegenerate matrix, namely: $\overline{\Psi}(r) = \sigma_2\Psi(r)$, $\det\sigma_2 \neq 0$. *The eigenvalue problem (10) or (11) is self adjoint*, provided appropriate boundary conditions are imposed. As such, it is sufficient to use the zero and periodic (or antiperiodic) boundary conditions at the ends of the interval.

We shall consider equations (10), (11) in four intervals: $(0, \infty)$, $(r_L, \infty)$, $(0, r_U)$, $(r_L, r_U)$, $r_L, r_U > 0$ depending on the regions—called *sectors*—within which the parameters $\varepsilon$ and $\alpha Z$ may lie, and pass to the limit $r_L \to 0$, $r_U \to \infty$ afterwards. So, in the end, all the intervals are one interval. The boundary conditions are

$$G(0) = G(\infty) = 0, \qquad \text{when} \qquad |\alpha Z| < 1, |\varepsilon| < m;$$
$$\text{the interval is} \qquad 0 \leqslant r \leqslant \infty \qquad \text{(sector I)} \tag{12}$$

$$G(0) = G(r_U) = 0, \qquad \text{when} \qquad |\alpha Z| < 1, |\varepsilon| > m;$$
$$\text{the interval is} \qquad 0 \leqslant r \leqslant r_U \qquad \text{(sector II)} \tag{13}$$

$$G(r_L) = G(\infty) = 0, \qquad \text{when} \qquad |\alpha Z| > 1, |\varepsilon| < m;$$
$$\text{the interval is} \qquad r_L \leqslant r \leqslant \infty \qquad \text{(sector III)} \tag{14}$$

$$G(r_L) = \pm G(r_U), \qquad F(r_L) = \pm F(r_U), \qquad \text{when} \qquad |\alpha Z| > 1, |\varepsilon| > m;$$
$$\text{the interval is} \qquad r_L \leqslant r \leqslant r_U \qquad \text{(sector IV)} \tag{15}$$

As long as $r_L \neq 0$ for $\alpha|Z| > 1$ and/or $r_U \neq \infty$ for $|\varepsilon| > m$, there are infinite countable manifolds of eigenvalues in sectors II, III, IV, which condense in the limit $r_L \to 0$, $r_U \to \infty$ to make continua.

The function $\varepsilon + \frac{\alpha Z}{r}$ does not change sign throughout the interval $(0, \infty)$, when $R > 0$, i.e., in the positive-energy domain ($\varepsilon > 0$) for positive charge $Z > 0$ and in the negative-energy domain ($\varepsilon < 0$) for negative charge $Z < 0$. In these regions, *the self-adjoint boundary problem is positively definite*. In this case, the eigenvalues are real and the corresponding solutions make, for each value of $R$, a complete set of states, mutually orthogonal with the measure

$$\mathrm{d}\mu(r) = \left(1 + \frac{R}{r}\right)\mathrm{d}r. \tag{16}$$

Below we shall also consider negative $R$ and see what happens in that case. Note that the $e^+ e^-$- pairs production just occurs in the region $R < 0$ of sector IV.

The domains, pointed out in (12)–(15), correspond to what was called sectors I–IV in [15, 14]. Sector II ($\alpha|Z| < 1$, $|\varepsilon| > m$) corresponds to particles, free at infinity in the limit $r_U \to \infty$ and repulsed from the centre. Only one of two fundamental solutions belongs to $L^2_\mu(0, r_U)$, the space of functions square integrable with the measure (16) in the interval $(0, r_U)$. Sector III ($\alpha|Z| > 1$, $|\varepsilon| < m$) contains particles, free in the origin in the limit $r_L \to 0$ and repulsed from the infinitely remote point. Only one fundamental solution belongs to $L^2_\mu(r_L, \infty)$). In sector IV ($\alpha|Z| > 1$, $|\varepsilon| > m$), particles are free both near the origin and near the infinity in the limit $r_L \to 0$, $r_U \to \infty$. The both fundamental solutions are $L^2_\mu(r_L, r_U)$. The (anti)periodic boundary conditions (15) are imposed in agreement with the inelastic character of the scattering in sector IV, where there is a nonzero current inflow through the outer border $r = r_U$, equal to the outflow through the inner border $r = r_L$ and vice versa. As distinct from this, in sectors II and III the current is zero, the scattering of particles emitted by the infinity (sector II) and by the centre (sector III) is elastic: everything emitted by the centre (infinity) is reflected back to the centre (infinity). As for sector I ($\alpha|Z| < 1, |\varepsilon| < m$), where the infinite countable manifold of hydrogen-like bound states lies in the quadrants ($0 < \alpha Z < 1, 0 < \varepsilon < m$) and ($-1 < \alpha Z < 0, -m < \varepsilon < 0$), neither fundamental solution generally belongs to $L^2_\mu(0, \infty)$.

To be more explicit, let us—following the standard way—left multiply equation (10) (or (11)) by the spinor $\Psi_1^*(r)$, which is the solution of the complex-conjugate equation, but with $\varepsilon_1$ taken instead of $\varepsilon$ (or $\alpha Z_1$ instead of $\alpha Z$) and the same $R$. (We stress that all the coefficients and matrices in expression (3) are real and do not depend either on $\varepsilon$ or on $\alpha Z$). Let us, next, subtract the same product, with $\Psi(r)$ and $\Psi_1^*(r)$ interchanged and integrate the difference over $r$ within the limits $r_1 < r < r_2$. In agreement with (12)–(15), in sectors I and II the lower limit should be chosen as $r_1 = 0$, whereas in sectors III and IV it is $r_1 = r_\mathrm{L}$. The upper limit $r_2$ coincides with $r_\mathrm{U}$ in sectors II and IV, and is infinite $r_2 = \infty$ in sectors I and III. Then one has

$$\begin{aligned} \mathrm{i}\Psi_1^*(r)\sigma_2\Psi(r)\big|_{r_1}^{r_2} &= (\varepsilon_1^* - \varepsilon)\int_{r_1}^{r_2} \Psi_1^*(r)\Psi(r)\left(1 + \frac{R}{r}\right)\mathrm{d}r \\ &= \alpha(Z_1^* - Z)\int_{r_1}^{r_2} \Psi_1^*(r)\Psi(r)\left(\frac{1}{R} + \frac{1}{r}\right)\mathrm{d}r. \end{aligned} \tag{17}$$

When written explicitly in components this equation looks like

$$\begin{aligned} (G_1^*(r)F(R) - F_1^*(r)G(r))|_{r_1}^{r_2} &= (\varepsilon_1^* - \varepsilon)\int_{r_1}^{r_2} (G(r)G_1^*(r) + F(r)F_1^*(r))\left(1 + \frac{R}{r}\right)\mathrm{d}r \\ &= \alpha(Z_1^* - Z)\int_{r_1}^{r_2} (G(r)G_1^*(r) + F(r)F_1^*(r))\left(\frac{1}{R} + \frac{1}{r}\right)\mathrm{d}r. \end{aligned} \tag{18}$$

The boundary conditions (12), (14), (13), (15) provide the vanishing of the left-hand side of (17) or (18) and make the differential expression $\mathcal{L}$ a self-adjoint operator. By omitting the index 1 in this relation, we find that for the eigenvalues $\varepsilon$ and $Z$ of the self-adjoint operator $\mathcal{L}$ to be real

$$\varepsilon = \varepsilon^*, \qquad Z = Z^* \tag{19}$$

it is sufficient that the function $\left(\frac{1}{R} + \frac{1}{r}\right)$ should have a definite sign, since in this case the integral in (17) is nonzero. Such a situation occurs when $R > 0$.

Otherwise, when $R < 0$, the zero-norm vectors, satisfying the relation

$$\int_{r_1}^{r_2} \Psi^*(r)\Psi(r)\left(1 + \frac{R}{r}\right)\mathrm{d}r = 0, \tag{20}$$

might exist and then the eigenvalues corresponding to it are not necessarily real. Also negative-norm solutions might appear with real eigenvalues, and the manifold of eigenfunctions of the self-adjoint operator (3) might not be complete, according to [20, 21]. In such a case it had to be completed to the Hilbert space, as explained in [22]. As a matter of fact, this scenario can be avoided.

It follows from (18), (12)–(15) that the eigenfunctions with nonzero norm obey the orthogonality relations

$$\int_{r_1}^{r_2} \Psi_k^*(r)\Psi_n(r)\left(1 + \frac{R}{r}\right)\mathrm{d}r = N\delta_{kn}. \tag{21}$$

Here, the integers $k, n$ label the eigenfunctions belonging to different countable eigenvalues $\varepsilon$ or $Z$, and the wavefunctions $\Psi$ relate to a fixed value of $R$. The norm $N$ in every sector, except sector I, gets predominant contribution from the integration near the points $r = r_\mathrm{L}$ and $r = r_\mathrm{U}$.

Bearing in mind the asymptotic behaviours (7) and (8), and the boundary conditions (15), we get for the eigenfunctions in sector IV in the asymptotic regime of very large upper size of the interval $r_{\mathrm{U}} \gg |R|, m^{-1}$ and very small lower size $r_{\mathrm{L}} \ll |R|, m^{-1}$—up to a finite factor—that

$$N^{\mathrm{IV}} \asymp \int_{r_{\mathrm{L}}}^{r_{\mathrm{U}}} \left(1 + \frac{R}{r}\right) \mathrm{d}r \asymp r_{\mathrm{U}} + R \ln \frac{r_0}{r_{\mathrm{L}}}. \tag{22}$$

Here, $r_0$ is an arbitrary positive-dimensional constant, $r_{\mathrm{L}} \ll r_0 \ll r_{\mathrm{U}}$. We neglected $r_{\mathrm{L}}$ as compared to $\left|R \ln \frac{r_0}{r_{\mathrm{L}}}\right|$ and $\left|R \ln \frac{r_{\mathrm{U}}}{r_0}\right|$ as compared to $r_{\mathrm{U}}$. A change of $r_0$ does not violate equation (22) within its accuracy.

For positive $R$ the norm (22) is positive. For negative $R$ the norm (22) *has a definite sign* for a definite value of $R$, provided either of the inequalities

$$\frac{r_{\mathrm{U}}}{|R|} > \ln \frac{r_0}{r_{\mathrm{L}}} \qquad \text{or} \qquad \frac{r_{\mathrm{U}}}{|R|} < \ln \frac{r_0}{r_{\mathrm{L}}} \tag{23}$$

is observed during the limiting process $r_{\mathrm{L}} \to 0, r_{\mathrm{U}} \to \infty$. By imposing this additional requirement, we avoid the appearance of vectors with different signs of the norm within the same set of eigenfunctions. The zero-norm vectors of state do not appear either. Thus, the problem of indefinite metric is eliminated in sector IV. In other sectors it does not appear.

In sector II, where the eigenfunctions are $L^2_\mu(0, r_{\mathrm{U}})$, the integration in (21) near $r = 0$ is convergent, while the upper limit gives a predominant contribution to (21) for $r_{\mathrm{U}} \gg |R|, m^{-1}$ due to the asymptotic behaviour (7)

$$N^{\mathrm{II}} \asymp \int^{r_{\mathrm{U}}} \left(1 + \frac{R}{r}\right) \mathrm{d}r \asymp r_{\mathrm{U}}. \tag{24}$$

This has a definite sign. In sector III, where the eigenfunction is $L^2_\mu(r_{\mathrm{L}}, \infty)$, the integration in (21) near the upper limit converges, but near the lower limit gives a predominant contribution for $r_{\mathrm{L}} \ll |R|, m^{-1}$ due to the asymptotic behaviour (8)

$$N^{\mathrm{III}} \asymp \int_{r_{\mathrm{L}}} \left(1 + \frac{R}{r}\right) \mathrm{d}r \asymp R \ln \frac{r_0}{r_{\mathrm{L}}}. \tag{25}$$

This has a definite sign for a given sign of $R$. As for sector I, the eigenvalue problems (10), (11) with the boundary conditions (12) does not have any solutions for $R < 0$: the hydrogen-like bound states only lie in the segments with $R > 0$ in this sector. In the latter region, for the known discrete values of the energy $\varepsilon$, depending on the charge $\alpha Z$, which provide the fulfilment of the boundary condition (12), the solution decreases as $\exp\{-r\sqrt{m^2 - \varepsilon^2}\}$ at $r \to \infty$ and as $r^{\sqrt{1-\alpha^2 Z^2}}$ at $r \to 0$. Correspondingly, the norm

$$N^{\mathrm{I}} \asymp \int_0^\infty \Psi^*(r)\Psi(r) \left(1 + \frac{R}{r}\right) \mathrm{d}r < \infty \tag{26}$$

converges in spite of the singularity in the measure.

It may be seen from the results of [3], where the Klein–Gordon equation was considered, that the corresponding measure would in that case be $\mathrm{d}\mu(r) = \left(\frac{1}{R} + \frac{1}{r}\right)^2 \mathrm{d}r$, which is positively definite, and hence the problem of indefinite metrics does not arise.

Manifolds of eigenvalues of the problem (10) (or (11)) may belong simultaneously to two or three sectors: to sectors I, II, IV, if $|R| < m^{-1}$, to sectors I, III, IV, if $|R| > m^{-1}$ and to sectors I, IV, if $|R| = m^{-1}$. Correspondingly, the resolution of the unity includes integration over continua of eigenvectors in all the sectors II–IV crossed by the straight line $R = \text{const}$ in the plane $(\varepsilon, \alpha Z)$, the states of both signs of the charge and energy being simultaneously involved, as well as summation over bound states in sector I for $R > 0$. To conform the boundary conditions (13) in sector II and (15) in sector IV, what is needed when $0 > R > -m^{-1}$, we have to choose the left inequality in (23), so that the signs of the norms be the same—positive—within one set of eigenvalues. Analogously, when $R < -m^{-1}$, we have to choose the right inequality in it, so that the boundary conditions (14) in sector III and (15) in sector IV might be in mutual agreement and the sign of the norm be common—negative in this case. At last, in the special case $R = -m^{-1}$ the problem is indifferent to the choice of the inequality sign in (23). The unrestricted growth of the norm (22) in the limit $r_{\rm L} \to 0$, $r_{\rm U} \to \infty$ gives the possibility to replace the right-hand side of (21) by $\delta$-function of $(\varepsilon_1 - \varepsilon)$ or of $(Z_1 - Z)$.

The integrals (22), (24), (25) play the role of effective sizes of the quantization boxes in the corresponding sectors. Once the spacings between the levels prior to the limiting transition $r_{\rm L} \to 0$ and/or $r_{\rm U} \to \infty$ are inversely proportional to these effective sizes (cf [14]), we conclude that the densities of states are essentially different in different sectors, which is important for physical processes occurring when a change of parameters causes transitions between sectors, as in heavy-ion collisions.

Besides the family of Hilbert spaces labelled by the ratio $R$ (9), we can point two more families. The standard one is associated with the representation of the Dirac equation (2) in the form of the eigenvalue problem for the Hamiltonian

$$\mathcal{H} = \mathcal{L} - \frac{\alpha Z}{r}, \tag{27}$$

the operator, which leaves aside the term $\varepsilon\Psi$ of the Dirac equation, singular near $r = \infty$, but includes all the terms singular in the origin

$$\mathcal{H}\Psi(r) = \varepsilon\Psi(r). \tag{28}$$

This eigenvalue problem is well defined within our procedure, which is standard in this case, when there is no singular attractiveness in the origin, i.e., for $\alpha|Z| < 1$. The families of eigenvectors, orthogonal for different values of $\varepsilon$ with the measure $\mathrm{d}r$, are labelled by the charge $-137 < Z < 137$ and consist for each charge of the wavefunctions corresponding to discrete bound states in sector I, $-m < \varepsilon < m$, and two continua of positive, $\varepsilon > m$, and negative, $\varepsilon < -m$, energies in sector II.

The other one is associated with the representation of the Dirac equation (2) in the form of the generalized eigenvalue problem for the operator $\mathcal{L} - \varepsilon$, which leaves aside the term $\frac{\alpha Z}{r}\Psi$ of the Dirac equation, singular near $r = 0$, but includes all the terms singular in the infinitely remote point,

$$(\mathcal{L} - \varepsilon)\Psi(r) = \frac{\alpha Z}{r}\Psi(r). \tag{29}$$

This eigenvalue problem is well defined within our procedure, when there is no singular attractiveness in infinitely remote point, i.e., for $|\varepsilon| < m$. The families of eigenvectors, orthogonal for different values of $Z$ with the measure $\frac{\alpha}{r}\,\mathrm{d}r$, are labelled by the energy $-m < \varepsilon < m$ and consist for each energy of the wavefunctions corresponding to discrete bound states in sector I, $-137 < Z < 137$, and two continua of positive, $Z > 137$, and negative, $Z < -137$, charges in sector III.

## 3. Dilated coordinate space

The generalized eigenvalue problem (10) (or (11)) can be formally reduced to the form of the standard eigenvalue problem like (28)

$$-\mathrm{i}\sigma_2 \frac{\mathrm{d}\widetilde{\Psi}(\xi)}{\mathrm{d}\xi} + \frac{r(\xi)}{r(\xi) + R}\left(\frac{-\sigma_1}{r(\xi)} + m\sigma_3\right)\widetilde{\Psi}(\xi) = \varepsilon\widetilde{\Psi}(\xi) \tag{30}$$

with the help of the dilatation of the variable

$$\xi(r) = \int_{r_0}^{r} \left(1 + \frac{R}{r}\right) \mathrm{d}r = R \ln \frac{r}{r_0} + r - r_0. \tag{31}$$

The positive-dimensional constant $r_0$ here is arbitrary.

If $R > 0$, the transformation (31) is a monotonous function of $r$, and the function $r(\xi)$, inverse to $\xi(r)$, is single-valued. In this case, the spinor in (30) should be understood as $\widetilde{\Psi}(\xi) = \Psi(r(\xi))$, and the interval, where this equation is defined, is $\xi \in (-\infty, \infty)$. The denominator in the second term of (30) is nonzero.

If, however, $R < 0$, the denominator turns to zero in the point $r = -R$, and also the function $r(\xi)$, inverse to $\xi(r)$, is two-valued. Denote the two branches of the inverse function as $r_\mathrm{a}(\xi)$ and $r_\mathrm{b}(\xi)$:

$$\begin{aligned} &0 < r_\mathrm{a}(\xi) < -R, &\quad& \xi_{\min} < \xi < \infty \\ &-R < r_\mathrm{b}(\xi) < \infty, &\quad& \xi_{\min} < \xi < \infty. \end{aligned} \tag{32}$$

Due to (31) the extremum condition $(\mathrm{d}\xi/\mathrm{d}r) = 0$ is satisfied in the point $r = -R$, hence $\xi_{\min} = \xi(-R)$. The value of $\xi_{\min}$ contains arbitrariness due to the arbitrariness of $r_0$. Its specific choice is not important. Now (30) becomes two equations

$$-\mathrm{i}\sigma_2 \frac{\mathrm{d}\widetilde{\Psi}_{\mathrm{a,b}}(\xi)}{\mathrm{d}\xi} + \frac{r_{\mathrm{a,b}}(\xi)}{r_{\mathrm{a,b}}(\xi) + R} \left(\frac{-\sigma_1}{r_{\mathrm{a,b}}(\xi)} + m\sigma_3\right) \widetilde{\Psi}_{\mathrm{a,b}}(\xi) = \varepsilon \widetilde{\Psi}_{\mathrm{a,b}}(\xi), \qquad \xi_{\min} \leqslant \xi \leqslant \infty \tag{33}$$

for the two spinors $\widetilde{\Psi}_{\mathrm{a,b}}(\xi) = \Psi(r_{\mathrm{a,b}}(\xi))$, defined in the same interval of $\xi$, but with two different functions $r_\mathrm{a}(\xi)$ and $r_\mathrm{b}(\xi)$ that coincide in the end-point $r_{\mathrm{a,b}}(\xi_{\min}) = -R$. Each of these equations has a singularity at the end of the interval $\xi = \xi_{\min}$. The two spinors are connected by the continuity relation

$$\widetilde{\Psi}_\mathrm{a}(-R) = \widetilde{\Psi}_\mathrm{b}(-R). \tag{34}$$

The current

$$j(r) = \mathrm{i}\Psi^*(r)\sigma_2\Psi(r) = F(r)G^*(r) - F^*(r)G(r) \tag{35}$$

is conserved, $(\mathrm{d}j/\mathrm{d}r) = 0$, on the interval $0 < r < \infty$ due to equation (10). Under the transformation (31) it turns into two currents $\widetilde{j}_{\mathrm{a,b}}(\xi) = j(r_{\mathrm{a,b}}(\xi)) = \mathrm{i}\widetilde{\Psi}^*_{\mathrm{a,b}}(\xi)\sigma_2\widetilde{\Psi}_{\mathrm{a,b}}(\xi)$ that both conserve, $(\mathrm{d}\widetilde{j}_{\mathrm{a,b}}/\mathrm{d}\xi) = 0$, as a consequence of equations (33), in the interval $\xi_{\min} < \xi < \infty$. Equation (34) guarantees the coincidence of the two currents in the singular end-point $\xi = \xi_{\min}$. We conclude that the current does not undergo a discontinuity when passing the singularity point in equations (33), once the latter are obtained from (10).

The transformation (31) maps the point $r = \infty$ to $\xi = \infty$ and the point $r = 0$ to either positive or negative infinity, depending on the sign of the ratio $R$ (9). In terms of the dilated variable $\xi$, the asymptotic behaviour (8)—up to the coordinate-independent factor $r_0^{\pm\mathrm{i}\gamma}$—becomes

$$\exp\left(\pm\mathrm{i}\xi\frac{\gamma}{R}\right), \qquad |\xi| \to \infty. \tag{36}$$

This looks like an ordinary free wave in the $\xi$-space with the *pseudo*momentum $(\gamma/R) = \pm\sqrt{\varepsilon^2 - R^{-2}}$. (For the special value $R = -m^{-1}$, noted above, the psedomomentum coincides with the momentum.) Under the transformation (31), the measure (16) turns into $\mathrm{d}\mu(r) = \mathrm{d}\xi$, and the orthogonality relation (21) now is

$$\int_{\xi(r_1)}^{\xi(r_2)} \widetilde{\Psi}_k^*(\xi)\widetilde{\Psi}_n(\xi)\,\mathrm{d}\xi = N\delta_{kn}. \tag{37}$$

The norm (22) in sector IV becomes

$$N^{\mathrm{IV}} = \int_{-\xi_{\mathrm{L}}}^{\xi_{\mathrm{U}}} \mathrm{d}\xi = \xi_{\mathrm{L}} + \xi_{\mathrm{U}}, \qquad \text{if} \quad R > 0$$

and

$$N^{\mathrm{IV}} \asymp \int_{r_{\mathrm{L}}}^{-R} \left(1 + \frac{R}{r_{\mathrm{a}}(\xi)}\right) \mathrm{d}r_{\mathrm{a}}(\xi) + \int_{-R}^{r_{\mathrm{U}}} \left(1 + \frac{R}{r_{\mathrm{b}}(\xi)}\right) \mathrm{d}r_{\mathrm{b}}(\xi)$$
$$= \int_{-\xi_{\mathrm{L}}}^{\xi_{\min}} \mathrm{d}\xi + \int_{\xi_{\min}}^{\xi_{\mathrm{U}}} \mathrm{d}\xi \asymp \xi_{\mathrm{L}} + \xi_{\mathrm{U}}, \qquad \text{if} \quad R < 0 \tag{38}$$

where $\xi_{\mathrm{U}} \equiv \xi(r_{\mathrm{U}}) \asymp r_{\mathrm{U}} \to \infty$ and $\xi_{\mathrm{L}} \equiv -\xi(r_{\mathrm{L}}) \asymp R\ln(r_0/r_{\mathrm{L}}) \to \operatorname{sgn}R \cdot \infty$. Thus, in the representation of the dilated variable $\xi$ it is explicit that eigenfunctions in sector IV are normalizable to $\delta$-function, when $\xi_{\mathrm{L}} + \xi_{\mathrm{U}}$ tends to *positive* infinity. This occurs when $R > 0$ and when $-m^{-1} < R < 0$. In the latter case $\xi_{\mathrm{L}} < 0$, but $\xi_{\mathrm{L}} + \xi_{\mathrm{U}} \to +\infty$, once the sign $>$ is chosen in (23), as explained in section 2. In contrast, for $R < -m^{-1}$, we agreed above to choose the $<$ sign in (23), so that $\xi_{\mathrm{L}} + \xi_{\mathrm{U}} \to -\infty$, and solutions are normalized to $-\delta(\varepsilon - \varepsilon_1)$. Then the total probabilities should be defined as norms taken with the minus sign. If, at last, $R = -m^{-1}$, either convention about the sign in (23) and, correspondingly, about the total probability does fit.

Analogously, the norm (25) in sector III is

$$N^{\mathrm{III}} \asymp \int_{-\xi_{\mathrm{L}}} \mathrm{d}\xi = \xi_{\mathrm{L}} \to \operatorname{sgn}R \cdot \infty. \tag{39}$$

This is normalizable to $\delta$-function for $R$ positive and to minus $\delta$-function for $R$ negative.

The contents of this section perfects the demonstration of complete symmetry between the inner and outer coordinate spaces, $r \to 0$ ($|\xi| \to \infty$) and $r \to \infty$ ($\xi \to \infty$).

## 4. Transmission and reflection coefficients

In sector IV, free particles emitted by an infinitely remote point (waves, incoming from $r = \infty$) are partially reflected backwards, but partially penetrate to the vicinity of the origin to become free near it (waves, outgoing to $r = 0$), in other words—absorbed by the centre. Also the inverse process may proceed: free particles emitted by the centre (waves, incoming from the origin $r = 0$) are partially reflected back to the centre, but partially escape to infinity (waves, outgoing to $r = \infty$ and 'absorbed by the remote point'). Thus, the singular differential equation is no longer a one-particle problem, but corresponds to an inelastic, two-channel problem, characterized by a $2 \times 2$ scattering matrix, the same as in [15, 14]. Below we shall find the off-diagonal $S$-matrix element squared—the transmission coefficient that is responsible for absorption of electrons by the supercritical nucleus, provided that the electron energy is above $m$, and for spontaneous electron–positron pair production, provided that the energy is below $-m$. The latter conclusion holds true within the concept of the Dirac sea—usually appealed to, when this effect is considered in the customary context (see the review [3], and the monographs [8]). The negative-energy free electron, which at $r = \infty$ belongs to the filled Dirac sea, after it is scattered off the point-like nucleus, partially transmits to the centre to become free near it and leaves an empty vacancy in its stead, interpreted as a positron, free at $r = \infty$. Unlike this scenario, the one of [3], [8] specifies that the electron near the nucleus is not free, but occupies the deepest bound state.

In the present section, we deal only with positive nucleus charge $Z$ and are in sector IV

$$\alpha Z > 1, \qquad |\varepsilon| > m. \tag{40}$$

Consider the fundamental solution to equation (2), which behaves as a single exponent near $r = 0$

$$\Psi(r)|_{r\to 0} \simeq (2pr)^{\mathrm{i}\gamma} \begin{pmatrix} \left(\frac{\mathrm{i}(\zeta+\gamma)}{1-\frac{\mathrm{i}m\zeta}{\varepsilon}} + 1\right)\left(\frac{\varepsilon}{m}+1\right)^{\frac{1}{2}} \\ -\left(\frac{\zeta+\gamma}{1-\frac{\mathrm{i}m\zeta}{\varepsilon}} + \mathrm{i}\right)\left(\frac{\varepsilon}{m}-1\right)^{\frac{1}{2}} \end{pmatrix}, \tag{41}$$

where

$$p = \sqrt{\varepsilon^2 - m^2}, \qquad \gamma = \sqrt{\alpha^2 Z^2 - 1}, \qquad \zeta = (\alpha Z\varepsilon)/p \tag{42}$$

are real quantities. Besides, $|\zeta| > \gamma$. The same solution behaves at $r = \infty$ as

$$\Psi(r)|_{r\to\infty} \simeq \mathrm{e}^{-\mathrm{i}pr}(2pr)^{-\mathrm{i}\zeta} \begin{pmatrix} \left(\frac{\varepsilon}{m}+1\right)^{\frac{1}{2}} \\ -\mathrm{i}\left(\frac{\varepsilon}{m}-1\right)^{\frac{1}{2}} \end{pmatrix} \mathcal{E} + \mathrm{e}^{\mathrm{i}pr}(2pr)^{\mathrm{i}\zeta} \begin{pmatrix} \left(\frac{\varepsilon}{m}+1\right)^{\frac{1}{2}} \\ \mathrm{i}\left(\frac{\varepsilon}{m}-1\right)^{\frac{1}{2}} \end{pmatrix} \mathcal{G}. \tag{43}$$

The coordinate-independent coefficients $\mathcal{E}$ and $\mathcal{G}$ can be determined using the analytical continuation $\sqrt{1-\alpha^2 Z^2} = \mathrm{i}\sqrt{\alpha^2 Z^2 - 1}$ of the known exact solution (see [1, 2]) into supercritical region $\alpha^2 Z^2 > 1$. These are

$$\mathcal{E} = \frac{\Gamma(2\mathrm{i}\gamma+1)\,\mathrm{e}^{-\frac{\pi}{2}(\zeta+\gamma)}}{\mathrm{i}(\gamma-\zeta)\Gamma(\mathrm{i}(\gamma-\zeta))}, \qquad \mathcal{G} = \frac{\Gamma(2\mathrm{i}\gamma+1)\,\mathrm{e}^{-\frac{\pi}{2}(\zeta-\gamma)}}{\left(1-\mathrm{i}\frac{m}{\varepsilon}\zeta\right)\Gamma(\mathrm{i}(\gamma+\zeta))}. \tag{44}$$

Calculating the values of the conserved current (35) near $r = 0$ and near $r = \infty$ and equalizing the two values with one another, we obtain the unitarity relation in the form of the identity

$$|\mathcal{G}|^2 = |\mathcal{E}|^2 + \frac{2\gamma}{\zeta-\gamma}, \tag{45}$$

easy to verify explicitly by the substitution of (44).

In the positive-energy domain, $\varepsilon > m$ $(R > 0)$, in accord with (36), the exponential in (41) $r^{\mathrm{i}\gamma} \simeq \exp\left(\mathrm{i}\xi\frac{\gamma}{R}\right)$ oscillates with the same sign of frequency as the second term in (43). Referring to the second term in (43) as to the wave, incoming from infinity, and to (41) as the wave transmitted to the centre, we have to normalize the incoming wave to unity. This reduces to division of equation (45) over $|\mathcal{G}|^2$ to give it the form

$$1 = \mathbb{R}_+ + \mathbb{T}_+, \tag{46}$$

with $\mathbb{R}_+$ and $\mathbb{T}_+$ being the reflection and transmission coefficients, respectively:

$$\mathbb{R}_+ = \frac{|\mathcal{E}|^2}{|\mathcal{G}|^2}, \qquad \mathbb{T}_+ = \frac{2\gamma}{(\zeta-\gamma)|\mathcal{G}|^2}, \qquad \varepsilon > m. \tag{47}$$

As $\zeta > \gamma$, when $\varepsilon > m$ , one has $\mathbb{T}_+ > 0$, hence $\mathbb{R}_+ < 1, \mathbb{T}_+ < 1$. Finally, the following analytical expression can be deduced for the transmission coefficient (47) from (44)

$$\mathbb{T}_+ = 1 - \frac{|\mathcal{E}|^2}{|\mathcal{G}|^2} = 1 - \mathrm{e}^{-2\pi\gamma}\frac{\sinh\pi(\zeta-\gamma)}{\sinh\pi(\zeta+\gamma)} = \frac{2\mathrm{e}^{2\pi\zeta}\,\mathrm{sh}\,2\pi\gamma}{\mathrm{e}^{2\pi(\zeta+\gamma)}-1}, \qquad \varepsilon > m. \tag{48}$$

For sufficiently large $\gamma$ this becomes

$$\mathbb{T}_+ = \frac{1}{1-\mathrm{e}^{-2\pi(\gamma+\zeta)}}.$$

Owing to the two-fold character of the transformation inverse to (31) in the negative-energy domain, the identification of the incoming and reflected waves is there different. Now that $\varepsilon < -m$ $(R < 0)$, in accord with (36), the exponential in (41) $r^{\mathrm{i}\gamma} \simeq \exp\left(\mathrm{i}\xi\frac{\gamma}{R}\right)$ oscillates with the same sign of frequency as the *first* term in (43), it is the *first* term that should be referred to as the wave, incoming from infinity, and the second one as the reflected wave. Now the normalization of the incoming wave to unity implies the division of equation (45) over $|\mathcal{E}|^2$. Then equation (45) acquires again the form (46)

$$1 = \mathbb{R}_- + \mathbb{T}_-, \tag{49}$$

but this time the reflection and transmission coefficients are

$$\mathbb{R}_- = \frac{|\mathcal{G}|^2}{|\mathcal{E}|^2}, \qquad \mathbb{T}_- = \frac{2\gamma}{(\gamma - \zeta)|\mathcal{E}|^2}, \qquad \varepsilon < -m. \tag{50}$$

In this domain $\zeta < -\gamma$, and one has $\mathbb{T}_- > 0$, hence $\mathbb{R}_- < 1, \mathbb{T}_- < 1$. Finally, the following analytical expression can be found for the transmission coefficient in the negative-energy domain

$$\begin{aligned} \mathbb{T}_- &= 1 - \frac{|\mathcal{G}|^2}{|\mathcal{E}|^2} = \frac{\mathbb{T}_+}{\mathbb{T}_+ - 1} = 1 - \mathrm{e}^{2\pi\gamma}\frac{\sinh\pi(\zeta+\gamma)}{\sinh\pi(\zeta-\gamma)} \\ &= \frac{1 - \mathrm{e}^{-4\pi\gamma}}{\mathrm{e}^{-2\pi(\zeta+\gamma)} - \mathrm{e}^{-4\pi\gamma}}, \qquad \varepsilon < -m. \end{aligned} \tag{51}$$

For sufficiently large $\gamma$ this is simplified to

$$\mathbb{T}_- = \mathrm{e}^{2\pi(\zeta+\gamma)}. \tag{52}$$

The latter expression is valid already for $\alpha Z - 1 \gg 1/32\pi^2$.

In our preliminary publication [10], we did not take into account the fact that the two-valuedness of the coordinate transformation, inverse to (31), affects the identification of incident and reflected waves, and took (47) as universal expressions for the reflection and transmission coefficients valid both in the positive—and negative—energy domains. The reflection coefficient $\mathbb{R}_+$ (47), when extended to $\varepsilon < -m$, becomes greater than unity, while the transmission coefficient $\mathbb{T}_+$ (47) becomes negative. Facing such situation, known as superradiation, we concluded that the spontaneous pair production is impossible due to the Pauli ban: electrons, taken from the Dirac sea, would have been increased in number after reflected off the nucleus. This is, however, forbidden, since all the vacancies in the Dirac sea are filled. We now renounce this point of view: the reflection and transmission coefficients in the negative-energy domain defined by (50) and (51) obey the inequalities $0 \leqslant \mathbb{R}_- \leqslant 1, 0 \leqslant \mathbb{T}_- \leqslant 1$, the same as (47), (48) in the positive-energy domain. In other words, the superradiation does not occur in our case of a spinor particle, where it might forbid the spontaneous pair creation.

The following comparison with the known results on scattering of Bose and Fermi particles off a black hole is in order. According to [16], if the black hole is rotating and correspondingly described by the Kerr metrics, the dilation transformation, analogous to (30) that reduces the corresponding propagation equation to a standard form of a Schrödinger-like equation may, in a certain kinematical domain, be two-fold reversible both for electromagnetic and gravitational waves, and the Dirac spin-$\frac{1}{2}$ field. Simultaneously, a singularity comes out from beyond the horizon into the coordinate region where the differential equation is defined. The general fact is that the current (or the Wronsky determinant) changes its sign when crossing this singularity for Bose fields, but does conserve for Fermi fields. (We observed the same property of current conservation across the singular point $r = -R$ in the differential equation (30) in section 3. Also this property holds true for the singular barrier problem associated with the second-order Schrödinger-like equation, to which the Dirac equation in the Coulomb potential was reduced in [3] and—after a coordinate-dilation transformation—in [10]. Stress that the singularity we are referring to is not the original singularity of the initial equation, but the one acquired in the course of its transformations.) Therefore, the superradiation takes place where there is the current discontinuity in the singularity point, i.e., for Bose particles.

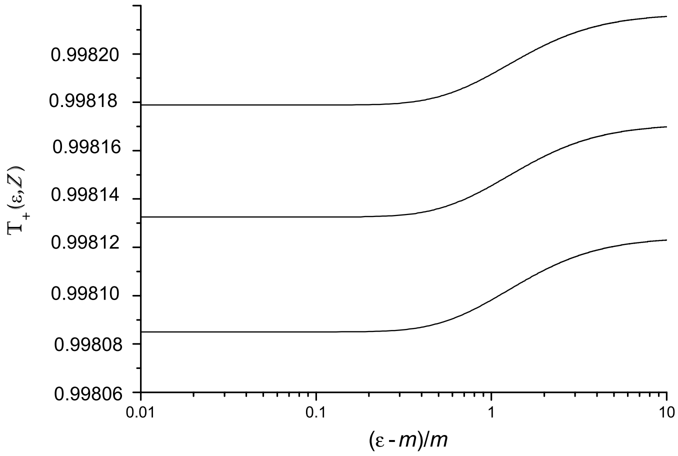


**Figure 1.** Transmission/absorption coefficient $\mathbb{T}_+(\varepsilon, Z)$ (48), (42) plotted in logarithmic scale against electron kinetic energy $\frac{\varepsilon - m}{m} > 0$ for three values of nucleus charge, from bottom to top, $\gamma = 0.498, 0.500, 0.502$ ($\alpha Z = 1.117, 1.118, 1.119$). The transmission/absorption coefficient is close to unity.

## 5. Absorption of electrons and production of electron–positron pairs

We now turn to more thorough considerations of expressions (48) and (51), derived above, and of their physical implementation.

The transmission coefficient (48), when multiplied by an electron flux density, is the absorption probability of electrons, incident on the point-like nucleus with $Z > 137$. Substituting (42) in (48), we get the transmission/absorption coefficient as a function of $\varepsilon$ and $Z$, $\mathbb{T}_+(\varepsilon, Z)$. It has the following asymptotic values:

$$\begin{aligned} &\mathbb{T}_+(\infty, Z) = 1 - \mathrm{e}^{-2\pi\gamma}\frac{\sinh\pi(\alpha Z - \gamma(Z))}{\sinh\pi(\alpha Z + \gamma(Z))} < 1, \\ &\mathbb{T}_+(m, Z) = 1 - \mathrm{e}^{-4\pi\gamma(Z)} < 1, \qquad \mathbb{T}_+(\infty, Z) > \mathbb{T}_+(m, Z), \end{aligned} \tag{53}$$

that depend on the nucleus charge $Z$ (see figure 1). Near the threshold of absorption $\gamma = 0$ ($\alpha Z = 1$), at the border of sector IV, these asymptotic values behave as

$$\begin{aligned} &\mathbb{T}_+(\infty, Z)|_{\alpha Z\to 1} \simeq 2\pi\sqrt{(\alpha Z)^2 - 1}(1 + \coth\pi) = 4.008\pi\sqrt{(\alpha Z)^2 - 1}, \\ &\mathbb{T}_+(m, Z)|_{\alpha Z\to 1} \simeq 4\pi\sqrt{(\alpha Z)^2 - 1}. \end{aligned} \tag{54}$$

This means that near the very threshold the absorption is very low. In contrast, already for $\alpha Z \simeq 1.12$ ($\gamma \simeq 0.5$) the value of $\mathbb{T}_+(\varepsilon, Z)$ exceeds 0.998 in the whole energy range $m < \varepsilon < \infty$ (see figures 1 and 2).

The threshold behaviour of (48) for any energy is

$$\mathbb{T}_+(\varepsilon, \gamma)|_{\alpha Z\to 1} \simeq 2\pi\sqrt{(\alpha Z)^2 - 1}\left(1 + \coth\frac{\pi\varepsilon}{\sqrt{\varepsilon^2 - m^2}}\right), \qquad \varepsilon > m. \tag{55}$$

It is also of interest to consider the transmission/absorption coefficient as a function of $\varepsilon$ and $R$, since the Hilbert space is formed by solutions with fixed ratio (9), according to section 2. Call $\widetilde{\mathbb{T}}_+(\varepsilon, R)$ the function, obtained from (48), by the substitution ($R > 0$, once $\varepsilon \geqslant m, \alpha Z \geqslant 1$)

$$\gamma = \sqrt{\varepsilon^2 R^2 - 1}, \qquad \zeta = \frac{\varepsilon^2 R}{\sqrt{\varepsilon^2 - m^2}}. \tag{56}$$

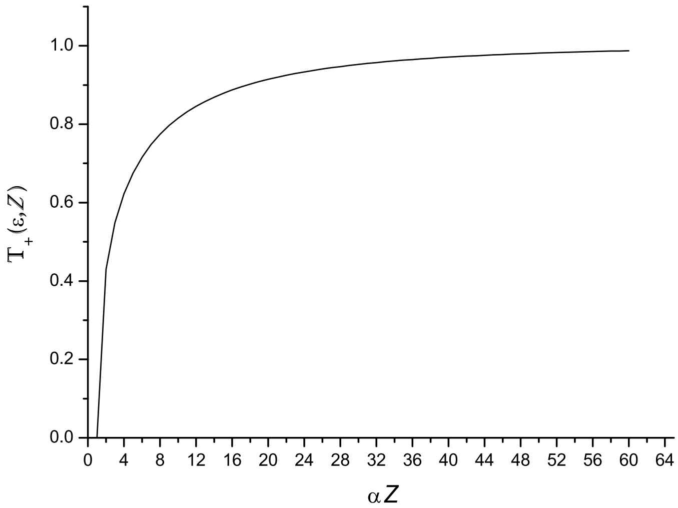


**Figure 2.** Transmission/absorption coefficient $\mathbb{T}_{+}(\varepsilon, Z)$ (48), (42) plotted against the nucleus charge for the electron kinetic energy value $\varepsilon - m = 11 \times 10^{-6}m$. The curves for other energy values are indistinguishable in the scale of the figure from the plotted one: the difference with the transmission/absorption coefficient for $\varepsilon - m = 100\,m$ is about $10^{-4}$.

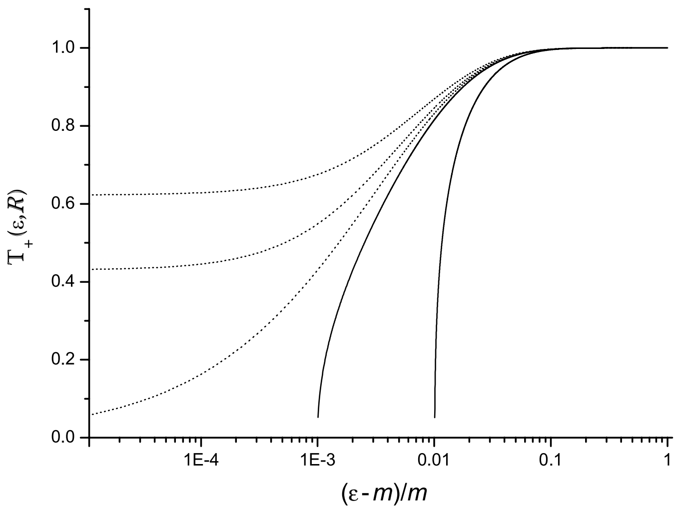


**Figure 3.** Transmission/absorption coefficient $\widetilde{\mathbb{T}}_{+}(\varepsilon, R)$ (48), (56) plotted in logarithmic scale against electron kinetic energy $\frac{\varepsilon-m}{m} > 0$ for five values of the ratio (9), from left to right, $Rm = 1.003, 1.001, 1.000, 0.999, 0.990$. The curves with $Rm < 1$ (drawn as solid) show the threshold behaviour (59).

Now the asymptotic value

$$\widetilde{\mathbb{T}}_{+}(\infty, R) = 1 \tag{57}$$

is universal for every $R$. A family of curves $\widetilde{\mathbb{T}}_{+}(\varepsilon, R)$ is shown in figure 3.

If $R \geqslant m^{-1}$, the border of sector IV is at $\varepsilon = m, \gamma \geqslant 0$. The values of $\widetilde{\mathbb{T}}_{+}(\varepsilon, R)$ at this border are

$$\widetilde{\mathbb{T}}_{+}(m, R) = 1 - \mathrm{e}^{-4\pi\sqrt{m^2R^2-1}}, \qquad R \geqslant m^{-1} \tag{58}$$

and correspond to crossings of the axis $\varepsilon = m$ with the family. (Note that the ordinate axis in figure 3 corresponds to $\varepsilon - m = 10^{-5}m$ and not to $\varepsilon - m = 0$.)

If $R \leqslant m^{-1}$, the border of sector IV is at $\varepsilon \geqslant m$, $\gamma = 0$. The curves in figure 3 cross the abscissa axis in the points $\varepsilon = \varepsilon_{\rm thr} \equiv R^{-1}$, found from the equation $\gamma = 0$. The threshold behaviour of the transmission/absorption coefficient near these points is

$$\widetilde{\mathbb{T}}_{+}(\varepsilon, R)|_{\varepsilon\to\varepsilon_{\rm thr}=R^{-1}} \simeq 2\pi(2R)^{1/2}\sqrt{\varepsilon - R^{-1}}\left(1 + \coth\frac{\pi}{\sqrt{1-m^2R^2}}\right), \qquad \varepsilon \geqslant R^{-1} > m. \tag{59}$$

The absorption takes place, for $R$ fixed, for electron energies exceeding the threshold values $\varepsilon_{\rm thr} = R^{-1}$.

In the negative-energy domain $\varepsilon \leqslant -m$ of sector IV, $\alpha Z > 1$, $R < 0$, the transmission coefficient (51), when multiplied by the (degenerate) Fermi distribution of electrons in the Dirac sea, which is unity, becomes the distribution of positrons, spontaneously produced from the vacuum. These distributions as functions of energy, $\widetilde{\mathbb{T}}_{-}(\varepsilon, R)$, are presented for the fixed negative ratio $R$ in figure 4, showing the transmission coefficient (51), with (56) substituted for $\gamma$ and $\zeta$ in it. The asymptotic value for large $|\varepsilon| \to \infty$

$$\widetilde{\mathbb{T}}_{-}(\varepsilon, R)|_{\varepsilon\to-\infty} \simeq \mathrm{e}^{2\pi(\zeta+\gamma)} \simeq \mathrm{e}^{-\frac{\pi m}{\varepsilon}\left(Rm+\frac{1}{Rm}\right)} \tag{60}$$

is equal to unity for every $R$:

$$\widetilde{\mathbb{T}}_{-}(-\infty, R) = 1, \qquad R < 0. \tag{61}$$

This means that the total probability of creating a positron with its energy less than infinity is 1.

Within the Dirac sea picture, we are sticking to, the mechanism, leading to the hole distributions like the ones drawn in figure 4, may be thought of as follows. According to section 2 (see the paragraph, preceding the one that includes equation (27)), the volumes of the inner and outer spaces unite in sector IV. Consequently, the number of states in it is larger than in sector II or III. The states in the negative-energy part of sector II are all occupied, whereas the newly added states are vacant. As a result, electrons in the Dirac sea rearrange: they tend to leave the energy states with larger $|\varepsilon|$ and keep to the surface of the sea $\varepsilon = -m$. The larger $|\varepsilon|$, the more holes there are left. Generally, the vacant states are filled with a certain probability, determined by the distribution (51) found above. We have to admit that an analogue of Hawking's 'temperature' is inherent in the supercritical nucleus (although the hole distribution is not black body), which prevents the degeneration of electron gas in the Dirac sea in sector IV.

The surface $\widetilde{\mathbb{T}}_{-}(\varepsilon, R)$ is a maximum for every energy at $R = -m^{-1}$. This fact is reflected in figure 4: the curve for $R = -m^{-1}$ occupies the extreme left position. The pair creation process runs most efficiently, when $R = -m^{-1}$, in other words, the distribution of produced particles is maximum at this value of $R$. The corresponding form of the distribution (51) is

$$\widetilde{\mathbb{T}}_{-}(\varepsilon, -m^{-1}) = \frac{1 - \mathrm{e}^{-4\pi\frac{p}{m}}}{\mathrm{e}^{2\pi\frac{m}{p}} - \mathrm{e}^{-4\pi\frac{p}{m}}}. \tag{62}$$

Not close to the threshold $p = 0$, this is especially simple:

$$\widetilde{\mathbb{T}}_{-}(\varepsilon, -m^{-1}) = \mathrm{e}^{-2\pi\frac{m}{p}}. \tag{63}$$

The same as in the positive-energy domain described above, the value $|R| = m^{-1}$ discriminates two different situations. If $|R| > m^{-1}$, $R < 0$, the border of sector IV is $\varepsilon = -m$, $\gamma \geqslant 0$, and the bordering value of the distribution function (51) is zero:

$$\widetilde{\mathbb{T}}_{-}(-m, R) = 0, \qquad |R| > m^{-1}, \tag{64}$$

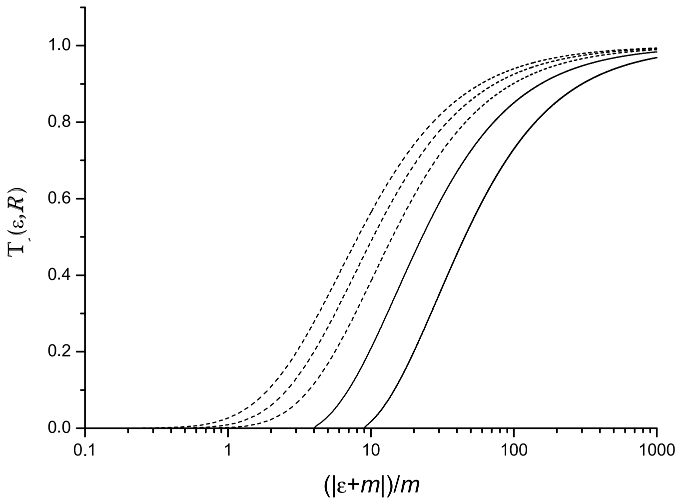


**Figure 4.** Transmission coefficient/positron distribution $\widetilde{\mathbb{T}}_{-}(\varepsilon, R)$ (51), (56) plotted in logarithmic scale against the negative energy $\frac{\varepsilon+m}{m} < 0$ for five negative values of the ratio (9), from left to right, $Rm = -1, -2, -3, -0.2, -0.1$. The curves with $|R|m < 1$ (drawn as solid) do not gather in the origin and show the threshold behaviour (65).

as is also seen in figure 4: the curves with $|R| > m^{-1}$ gather in the origin $\widetilde{\mathbb{T}}_{-} = 0, \varepsilon + m = 0$. If $|R| < m^{-1}$, $R < 0$, the border of sector IV is $\gamma = 0$, $\varepsilon < -m$. The threshold behaviour of the positron distribution is

$$\widetilde{\mathbb{T}}_{-}(\varepsilon, R)|_{\varepsilon\to\varepsilon_{\rm thr}=R^{-1}} \simeq \pi(2)^{3/2}\sqrt{\varepsilon R - 1}\left(1 + \coth\frac{\pi}{\sqrt{1-m^2R^2}}\right), \qquad \varepsilon \leqslant R^{-1} < -m. \tag{65}$$

This is zero in the threshold points $\varepsilon_{\rm thr} = R^{-1}$, which are the intersections between the curves with $R > -m^{-1}$ and the abscissa axis in figure 4.

By differentiating (65) over $\varepsilon$ we obtain that the differential probability is singular,

$$\sim(\varepsilon - R^{-1})^{-1/2}, \tag{66}$$

near the threshold $\varepsilon = \varepsilon_{\rm thr} = R^{-1}$ of the positron production process. This singularity may be referred to as a resonance in the sense that its appearance occurs when the energy is strictly equal to its bordering value, namely $\varepsilon = R^{-1} < 0$, which is the point where the line $R = \text{const} > -m^{-1}$ crosses the border $\alpha Z = 1$ between two continua of sectors II and IV. The corresponding singularities are seen as infinite peaks against the background of gentler and wider maxima in figure 5.

The following heuristic explanation of why the singular threshold behaviour in positron production probability takes place at the border between sectors II and IV, and not between sectors III and IV, may be proposed. According to the mechanism of the positron distribution formation, described above in this section, when we cross the border $\alpha Z = 1, \varepsilon < -m$ between sectors II and IV at the energy value below the surface of the Dirac sea, we get at once into the region where the ground state (the Dirac sea) is rearranged and the hole distribution is already different from zero, note the abrupt growth of the distribution with the energy behind the threshold in (65). This relates to the solid lines in figures 4 and 5. In contrast, when we cross the border $\alpha Z > 1, \varepsilon = -m$ between sectors III and IV, we get into sector IV just on the surface of the Dirac sea where the occupation number for holes remains zero. This relates

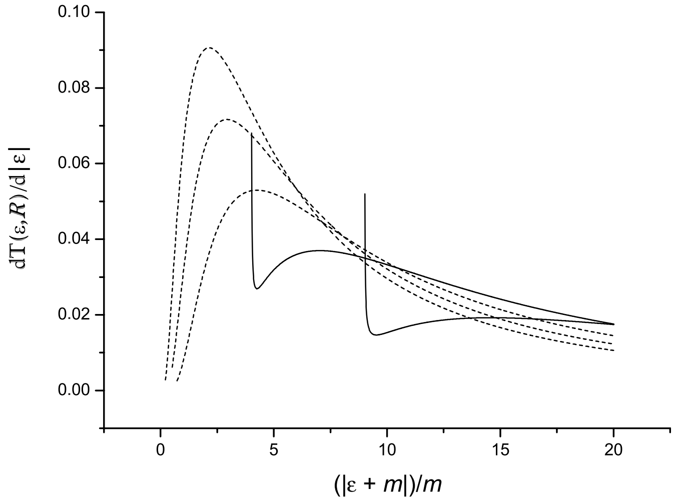


**Figure 5.** Differential probability of positron creation—derivative of the transmission coefficient (51), (56) $\mathrm{d}\widetilde{\mathbb{T}}_{-}(\varepsilon, R)/\mathrm{d}|\varepsilon|$ plotted against the negative energy $\frac{\varepsilon+m}{m} < 0$ for five negative values of the ratio (9), from left to right, $Rm = -1, -2, -3, -0.2, -0.1$. The curves with $|R|m < 1$ (drawn as solid) show the resonant threshold behaviour (66).

to the gradual growth of the distribution (51), starting with the zero value (64),

$$\widetilde{\mathbb{T}}_{-}(\varepsilon, R)|_{\varepsilon\to -m^2-0} \simeq \mathrm{e}^{\frac{2m^2 R}{\sqrt{\varepsilon^2-m^2}}} \coth 2\pi\sqrt{m^2R^2-1} \to 0, \qquad R < 0, \qquad m|R| > 1. \tag{67}$$

as reflected by the dotted lines in figures 4 and 5.

The fact that the probabilities turn to infinity in the resonance points (remind that the distributions (65) remain finite there) does not make a physical difficulty, since the energy is ever measured only with a certain accuracy.

The inverse-square-root singularity in the threshold behaviour is encountered in other physical problems as well. It is characteristic, e.g., of the single $e^+e^-$-pair creation amplitude by a photon in a constant magnetic field [23] or of the polarization tensor (dielectric permeability) of a photon in such field [24]. In this case, the singular threshold behaviour is due to the fact that real or virtual electron and positron, created on definite Landau levels, possess only one continuous degree of freedom, responsible for the motion along the magnetic field. This cyclotron resonance in the vacuum polarization has quite real physical implementations and introduces the effect of photon capture by a strong magnetic field in the vacuum [25], in semiconductors [26] and in relativistic electron–positron plasma [27]. It completely determines [28] the refraction index for magnetic fields much larger than the Schwinger ‘critical’ value $B_{\mathrm{cr}} = (m^2/e) = 4.4 \times 10^{13}$ G.

Note that the condition $|R| < m^{-1}$ for the resonance to take place in the pair-production differential probability coincides with the condition of the positivity of the norm discussed in section 2.

## 6. Discussion

The existing theory of spontaneous $e^+e^-$-pair creation by a supercharged nucleus [3, 8] does not solve the problem of singular interaction, but is satisfied with the pragmatically sufficient view that the realistic nucleus has a finite size, which provides the regularizing cut-off of the potential near $r = 0$. The size of the nucleus core is taken to be about $10^{-12}$ cm, which is an order of magnitude less than the only characteristic size in the problem, the electron Compton wavelength $m^{-1} = 3.9 \times 10^{-11}$cm. Consequently, all the results in that theory—including the value of the critical charge itself—are cut-off dependent and do not survive its removal. Besides, as distinct from ours, those are based on numerical calculations. These circumstances forbid a direct comparison, because in our approach no cut-off is kept. Nevertheless, we can point a certain correspondence between the spontaneous pair creation curves.

The theory of [3, 8] points to the critical value of the nucleus charge $Z$, the one for which the lowest electron level first sinks into the lower continuum $\varepsilon < -m$ and becomes a resonance, or a quasistationary state. The corresponding scattering amplitude acquires the Breit–Wigner shape. By considering a nonstationary problem, where the nucleus charge crosses its critical value in the course of time evolution, one establishes—with the use of the Fano theory [8]—that the decay probability of the quasistationary state is determined by the same Breit–Wigner function. The quasistationary state decays into a pair: the electron, which becomes a bound state localized near the centre, and the hole, which escapes to infinity and is interpreted as a free positron, subject to observation. (In our treatment, the created electron belongs to continuum of states, free near the centre.) Thus, characteristic of the differential probabilities, obtained in this way, is a narrow Breit–Wigner resonance peak, its width depending on the cut-off. One may think that the singular threshold behaviour seen in figure 5 may be substituting for this resonance.

As for the electron absorption by the supercritical nucleus, described in section 5 and in [10], this effect is unknown to the conventional theory. According to figures 1 and 2, the absorption runs very efficiently. It might be observed if the heavy-ion collision process is subjected to irradiation by a beam of electrons or if the electrons produced in this process itself via, e.g., the two-photon mechanism are absorbed.

One may wonder how our method works in the realistic case of a finite-size nucleus, to which the point-like nucleus serves as an idealization, and how we may justify the literal discrepancy—although similarity—with the results of the conventional procedure, which is seemingly unambiguous in this case. The natural prescription within our approach is to take the size of the nucleus $r_{\rm N}$ as the lower edge of the regularizing box, introduced in section 2, $r_{\rm N} = r_{\rm L}$. (We stress that this 'regularization' is different from the cut-off in [3, 8]). With $r_{\rm N} \ll m^{-1}$, the asymptotic behaviour (8) in (41) is already reached, when $r \to r_{\rm N}$ and the momentum $p$ is not much greater than several electron masses. This means that our results for transmission coefficients remain unchanged, only become related to restricted region of energy. This nucleus-size independence is analogous to the well-known fact that scattering amplitudes do not depend upon the size of the laboratory (in our context, on the upper end of the box $r_{\rm U}$), provided the latter is much greater than any characteristic length parameter in the problem. The value of the Coulomb electric field intensity $E = eZ/r^2$ at the edge of a sphere, whose radius $r$ is the Compton length $r = m^{-1}$ and the charge is $Z = e^{-2} = 137$, coincides with the Schwinger 'critical' value $E_{\rm cr} = m^2/e$, which makes about $10^{16}$ V cm$^{-1}$. If the sphere radius is $r_{\rm N} \sim 10^{-12}$ cm, this value is two orders of magnitude larger. It can be admitted that when the external field is that large (also its gradient is large) there may exist deviation from the predictions of conventional theory.

We may conclude by saying that if electric charge greater than 137 is compressed to a sphere with its radius less than the electron Compton wavelength, it starts behaving as an absorbing/emitting, black-hole-like object, to which the present approach is applicable, its predictions concerning absorption/emission amplitudes being independent of the radius.

To let our approach be matched gradually with the conventional one when the nucleus radius is taken as larger, we may propose, for instance, the generalized eigenvalue problem, which is determined by the following form of the Dirac equation (2):

$$\left(\mathcal{L} - \frac{mr_{\mathrm{N}}}{1 + mr_{\mathrm{N}}} V_{r_{\mathrm{N}}}(r)\right) \Psi(r) = \left(\varepsilon + \frac{1}{mr_{\mathrm{N}} + 1} V_{r_{\mathrm{N}}}\right) \Psi(r), \tag{68}$$

where the regularized potential $V_{r_{\mathrm{N}}}(r)$ tends to $eZ/r$ when $r_{\mathrm{N}} \to 0$. In the limit $r_{\mathrm{N}} \ll m^{-1}$, the form (68) coincides with (2), but for large $r_{\mathrm{N}} \gg m^{-1}$ it becomes the standard eigenvalue problem $H\Psi = \varepsilon\Psi$.

There is another point worth discussing. In our treatment above, the same as in the traditional treatment of the pair-production process by a supercharged nucleus [3, 8], the Dirac sea concept of the ground state and the hole theory of positrons were appealed to. It is known, however, that this concept cannot be done completely consistently, since the presence of a charge in the ground state remains not excluded within its scope. Nevertheless, this concept possesses a certain predictive power and invokes useful analogies taken from the solid-state physics and theory of phase transitions. The interpretation of the produced positron distribution as resulting from the adding of a manifold of vacant states to the Dirac sea, proposed in section 5 above, may be referred to as an example of this sort.

Meanwhile, the approach, developed in the present paper—see particularly section 3, where the singular problem is reduced to the barrier transmission/reflection on an infinite axis—suggests a natural context for applying the theory of unstable vacuum, used in the literature to consider the Schwinger effect, i.e., particle production by an external electromagnetic field [18, 19], the Hawking radiation and Unruh's acceleration radiation [17]. This theory is based on second quantization and refers to nonequivalent Fock spaces, interrelated by the Bogoliubov transformation, the Bogoliubov coefficients being just the coefficients, that tangle asymptotes with positive and negative signs of momenta (in our case also of pseudomomenta) of the solutions to the equations of motion, prior to the second quantization. Leaving the detailed elaboration of the second-quantization programme of the singular problem to the forthcoming paper, we now restrict ourselves to formulating a clue point, important for the description of the electron–positron pair production.

Let $a^{\pm}_{\mathrm{out}}$ be operators, creating or annihilating states, which possess the wavefunction that behaves as one of the exponents (7) at the spatial infinity $r \to \infty$. Besides the Fock space, spanned by the vectors created by repeatedly applying $a^{+}_{\mathrm{out}}$ to the out-vacuum $|0_{\mathrm{out}}\rangle$, $a^{-}_{\mathrm{out}}|0_{\mathrm{out}}\rangle = 0$, define another Fock space, produced by the action on the true-vacuum $|0_{\mathrm{true}}\rangle$, $a^{-}_{\mathrm{true}}|0_{\mathrm{true}}\rangle = 0$, of the operator $a^{+}_{\mathrm{true}}$, which creates states, whose wavefunction is given by the solution of the generalized eigenvalue problem, specified in section 2. This eigenfunction behaves as a linear combination of the two exponents (7) at the spatial infinity $r \to \infty$ (also a linear combination of the two exponents (8) or (36) in the origin $r \to 0$ or $\xi \to \pm\infty$)—cf [14], where analogous eigenproblem was explicitly solved for the Schrödinger equation with singular potential. To determine the mean number of particles in the true-vacuum state, suffices it to calculate the true-vacuum expectation value of the particle-number operator $\langle 0_{\mathrm{true}}|a^{\dagger-}_{\mathrm{out}} a^{+}_{\mathrm{out}}|0_{\mathrm{true}}\rangle$. This is expressed in terms of the coefficients in the linear combination of the exponents mentioned and corresponds to the count of particles by a remote observer. Simultaneously, another observer, who is placed in the origin—or is infinitely remote in the dilated coordinate $\xi$-space, would measure the mean number of particles $\langle 0_{\mathrm{true}}|a^{\dagger-}_{\mathrm{in}} a^{+}_{\mathrm{in}}|0_{\mathrm{true}}\rangle$, where $a^{\pm}_{\mathrm{in}}$ are operators, creating or annihilating states with the wavefunction that behaves as one of the exponents (8) or (36) in the origin $r \to 0$ or $\xi \to \pm\infty$. The gases of particles, observed by the two observers, should be in mutual balance. The idea of balance is formally introduced into the theory when we impose the non-Sturm boundary conditions (15) in the eigenvalue problem that interrelate the values of the wavefunction in the points $r = 0$ and $r = \infty$.

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